

# OPERATOR SPLITTING FOR A HOMOGENEOUS EMBEDDING OF THE LINEAR COMPLEMENTARITY PROBLEM\*

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**Abstract.** We present a first-order quadratic cone programming algorithm that can scale to very large problem sizes and produce modest accuracy solutions quickly. Our algorithm returns primal-dual optimal solutions when available or certificates of infeasibility otherwise. It is derived by applying Douglas–Rachford splitting to a homogeneous embedding of the linear complementarity problem, which is a general set membership problem that includes quadratic cone programs (QCPs) as a special case. Each iteration of our procedure requires projecting onto a convex cone and solving a linear system with a fixed coefficient matrix. If a sequence of related problems are solved, then the procedure can easily be warm-started and make use of factorization caching of the linear system. We demonstrate on a range of public and synthetic datasets that for feasible problems our approach tends to be somewhat faster than applying operator splitting directly to the QCP, and in cases of infeasibility our approach can be significantly faster than alternative approaches based on diverging iterates. The algorithm we describe has been implemented in C and is available open-source in the solver SCS v3.0.

**Key words.** quadratic programming, cone programming, complementarity problems, monotone operators, operator splitting, Douglas–Rachford splitting, ADMM, first-order methods, homogeneous embeddings

**AMS subject classifications.** 49M05, 49M29, 65K05, 65K10, 90C05, 90C06, 90C20, 90C22, 90C25, 90C30, 90C33, 90C46

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**1. Introduction.** The goal in a linear complementarity problem (LCP) is to find a point in a convex cone that satisfies a complementarity condition [15, 22, 49]. In this paper we apply Douglas–Rachford (DR) splitting to a homogeneous embedding of the monotone LCP that encodes both the feasibility and infeasibility conditions of the original problem. Although the algorithm we develop is for general monotone LCPs, in this manuscript we focus on convex quadratic cone programs (QCPs), which are a special case. QCPs are a type of convex optimization problem where the goal is to minimize a quadratic objective subject to a conic constraint.

The recent SCS (splitting conic solver) algorithm is a first-order optimization procedure that can solve large convex *linear* cone problems to modest accuracy quickly [56, 57]. It is based on applying the alternating directions method of multipliers (ADMM) to a homogeneous self-dual embedding of the problem [9, 60, 79]. However, it cannot handle quadratic objectives directly, relying instead on reductions to second-order cone constraints. This reduction is inefficient in three ways. First, it is costly to perform the necessary matrix factorization required for conversion; second, the factorization may destroy any favorable sparsity in the original data; and third, it appears that operator splitting methods like ADMM are better able to exploit the strong convexity of a quadratic objective when used directly, rather than as a second-order cone [28, 48]. This limitation, and the myriad real-world applications with quadratic objectives, has inspired the development of first-order ADMM based solvers that tackle the quadratic objective directly [59, 68, 26]. However, solvers not based on

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a homogeneous embedding must rely on an alternative procedure based on diverging iterates to generate certificates of infeasibility if the problem does not have a solution [5, 38, 33, 6, 4, 3]. This procedure tends to be slower and less robust in practice [45]. In this paper we derive an algorithm that enjoys both properties—direct handling of quadratic objectives and efficient generation of infeasibility certificates.

Building on the homogeneous self-dual model of Goldman and Tucker [29], a series of papers developed homogeneous embeddings for the LCP [77, 76] and the more general monotone complementarity problem (MCP) [2]. Here we use the embedding of Andersen and Ye [2] applied to a monotone LCP. We show that the operator corresponding to the embedding is monotone, but not maximal, a property required for convergence of most operator splitting techniques. We derive a natural maximal extension of the operator which defines the final embedding. The resulting embedded problem can be expressed as finding a zero of the sum of two maximal monotone operators, to which we can apply standard operator splitting methods [7, 65].

We focus our attention on DR splitting due to its general convergence guarantees and good empirical performance [20, 37], though there are many alternative approaches [72, 61]. DR splitting is equivalent to ADMM under a particular change of variables [25, 23] (and indeed both are instantiations of the proximal point method [64]), and so the final method we derive is closely related to the SCS algorithm. Applying DR splitting to the embedded problem results in an iterative procedure with a per-iteration cost that is almost identical to the linear-convex case as tackled by SCS and to applying the splitting method directly to the original problem [75, 59].

There are several advantages that the homogeneous embedding approach has over competing methods of generating certificates of infeasibility based on diverging sequences [5, 38]. When using the homogeneous embedding, infeasibility certificates are generated by *convergence*. Alternative methods generate certificates by *divergence*, typically by examining the difference between successive iterates. This means when using the homogeneous embedding we have much more flexibility about how we converge to a solution. For instance, we can apply any procedure that guarantees convergence to a (nonzero) fixed point, which means we can use inexact or stochastic updates [64, 23], modern acceleration techniques [30, 78, 69, 66], or second-order extensions [1]. Moreover, approaches relying on DR splitting automatically benefit from the guaranteed  $o(1/k)$  bound on the convergence rate [32, 16]. This is in contrast to the difference of diverging iterates produced by DR splitting which have no guaranteed *rate* of convergence in general, satisfying a weaker notion of convergence instead [38, Thm. 3]. This stronger convergence guarantee is not just theoretical, since algorithms for the homogeneous embedding tends to be faster and more robust at detecting infeasibility in practice. This was shown recently for interior point methods [45] and we shall show similar results experimentally for DR splitting. On the other hand, if the problem is feasible, then using the homogeneous embedding does not appear to harm convergence when compared to tackling the original problem directly. On the contrary, we present numerical evidence to suggest that the homogeneous embedding approach can actually converge to a solution slightly faster than direct approaches even when the problem is feasible, at least when DR splitting is used.

QCPs are an important problem type with many applications, some of which we list here. Every linear program, quadratic program, second-order cone program, semidefinite program, exponential cone program, etc., can be formulated as a QCP. Sequential quadratic programming is an effective nonlinear constrained optimization algorithm that relies on solving a sequence of QCPs in order to converge to the solution of the original nonlinear problem [55, Chap. 18], [71]. In machine learning the

support vector machine and the lasso can both be formulated as QCPs [53, 70]. In portfolio optimization the standard trade-off between return and risk is often formulated as a QCP once additional constraints, such as trading costs, leverage limits, etc., are incorporated [43, 11]. Model predictive control with quadratic stage costs is a QCP with a particular sparsity structure [13, 59, 67]. Quadratic objectives over the semidefinite cone come up when solving matrix reconstruction and low rank matrix completion problems, where the goal is to find a positive semidefinite matrix with low rank that minimizes the Frobenius norm to some reference [34, 35]. Fast and robust generation of certificates of infeasibility is important in a range of applications. For example, in a branch and bound procedure applied to a mixed-integer quadratic programming problem some branches are infeasible and pruning those away reliably is crucial for good performance [36, 50].

*Software.* The quadratic cone programming algorithm we describe in this paper has been implemented in C and is available online at <https://github.com/cvxgrp/scs/tree/3.0.0>. It is written as an extension of the SCS solver and it thus inherits the capabilities of SCS. Specifically, it can solve convex QCPs involving any combination of nonnegative, second-order, semidefinite, exponential, and power cones (and their duals). It has multithreaded and single-threaded versions, can run on both CPU and GPU, and solves the linear system at each iteration using either a direct method or an iterative method. It can be used in other C, C++, Python, MATLAB, R, Julia, and Ruby programs and is a supported solver in parser-solvers CVX [31], CVXPY [18], Convex.jl [73], JuMP [21], and YALMIP [39].

**2. Monotone operator preliminaries.** This manuscript is concerned with operator splitting algorithms applied to a monotone inclusion problem, so here we cover the basic concepts that we use later; for more detail see, e.g., [7, 65]. An operator (or relation, point-to-set mapping, multivalued function)  $F$  on  $\mathbb{R}^d$  can be characterized by its graph, which is a subset of  $\mathbb{R}^d \times \mathbb{R}^d$  [7, Chap. 1.2]. We shall use the notation  $F(x)$  to refer to the set  $\{y \mid (x, y) \in F\}$  (we are abusing notation here by using  $F$  to refer to both the operator and its graph). Many of the operators we consider in this paper are single-valued, i.e., for a fixed  $x \in \mathbb{R}^d$  the set  $\{y \mid (x, y) \in F\}$  is a singleton and with some abuse of notation we shall write  $y = F(x)$  in this case.

An operator  $F$  is *monotone* if it satisfies

$$(u - v)^\top (x - y) \geq 0 \quad \forall (x, u), (y, v) \in F,$$

or in shorthand notation

$$(F(x) - F(z))^\top (x - z) \geq 0,$$

for all  $x, z \in \mathbf{dom}(F)$ , where the domain is taken to be  $\mathbf{dom}(F) = \{x \mid F(x) \neq \emptyset\}$ .

A monotone operator is *maximal* if it is not strictly contained by another monotone operator, i.e., extending  $F$  to include  $(x, u) \in \mathbb{R}^d \times \mathbb{R}^d$  would result in a non-monotone operator for any  $(x, u)$  not already in  $F$ . Maximality is not just a technical detail; it is an important property for convergence of the algorithms we develop in this manuscript and we shall verify that the operators we present are maximal monotone. Examples of maximal monotone operators include the identity operator  $I = \{(x, x) \mid x \in \mathbb{R}^d\}$  and the subdifferential  $\partial f = \{g \mid f(z) \geq f(x) + g^\top(z - x) \text{ for all } z \in \mathbb{R}^n\}$  of closed, convex, proper function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  [65].

**2.1. Operator splitting.** In this manuscript we deal with monotone inclusion problems involving the sum of two maximal monotone operators; that is, we want to find a  $u \in \mathbb{R}^d$  such that

$$(2.1) \quad 0 \in F(u) + G(u),$$

where  $F$  and  $G$  are maximal monotone operators on  $\mathbb{R}^d$ . Operator splitting methods are a family of algorithms for finding a zero in this case whereby we make use of the operators that define the problem separately. In this manuscript we focus on the well-known DR splitting method. DR splitting applied to the inclusion problem (2.1) is the following iterative procedure: From any initial  $w^0 \in \mathbb{R}^d$  repeat for  $k = 0, 1, \dots$ ,

$$(2.2) \quad \begin{aligned} \tilde{u}^{k+1} &= (I + F)^{-1}w^k, \\ u^{k+1} &= (I + G)^{-1}(2\tilde{u}^{k+1} - w^k), \\ w^{k+1} &= w^k + u^{k+1} - \tilde{u}^{k+1}. \end{aligned}$$

If a solution to (2.1) exists, then the DR splitting procedure generates a sequence of iterates  $(w^k, u^k, \tilde{u}^k)$  that satisfy  $\|u^k - \tilde{u}^k\| \rightarrow 0$ ,  $u^k \rightarrow u^*$ , and  $w^k \rightarrow w^* \in u^* + F(u^*)$ , where  $u^* \in \mathbb{R}^d$  is a solution [7, Thm. 26.11]. The quantity  $\|w^{k+1} - w^k\|_2^2$  converges to zero at a rate of  $o(1/k)$  [16, Cor. 2], [32, Thm. 3.1]. If a solution does not exist, then the iterates generated by DR splitting will not converge.

**2.2. Resolvent operator.** The first two steps of DR splitting require the evaluation of the *resolvent* of the two operators in the inclusion, which for operator  $F$  is  $(I + F)^{-1}$ . The resolvent of a maximal monotone operator is always single-valued, even if the operator that defines it is not, and has full domain [46, 47]. If  $F$  is the subdifferential of a convex function  $f$ , then the resolvent is known as the *proximal* operator [60] and is given by

$$(2.3) \quad \begin{aligned} y &= (I + \partial f)^{-1}x \\ \Leftrightarrow 0 &\in \partial f(y) + y - x \\ \Leftrightarrow y &= \operatorname{argmin}_z (f(z) + (1/2)\|z - x\|_2^2). \end{aligned}$$

**3. The monotone and linear complementarity problems.** QCPs are the main problems of interest in this paper and in this section we review the relationship between QCPs and LCPs, which are themselves a special case of MCPs. We introduce these complementarity problems and show their equivalence to monotone *inclusion* problems, to which we can apply operator splitting techniques. In what follows we shall embed the conditions for feasibility and infeasibility of an LCP into an MCP.

The monotone complementarity problem  $\text{MCP}(F, \mathcal{C})$  defined by maximal monotone operator  $F$  on  $\mathbb{R}^d$  and nonempty, closed, convex cone  $\mathcal{C}$  is to find a point  $z \in \mathbb{R}^d$  for which

$$(3.1) \quad \exists w \in F(z) \text{ s.t. } \mathcal{C} \ni z \perp w \in \mathcal{C}^*,$$

where  $\mathcal{C}^*$  denotes the dual cone to  $\mathcal{C}$ , i.e.,  $\mathcal{C}^* = \{w \mid w^\top z \geq 0, z \in \mathcal{C}\}$ . That is, the problem is to find a  $z \in \mathcal{C}$  such that for some  $w \in F(z) \cap \mathcal{C}^*$  we have  $z^\top w = 0$ . If  $F$  is single-valued, then we can write the problem more succinctly as finding a  $z \in \mathbb{R}^d$  such that  $\mathcal{C} \ni z \perp F(z) \in \mathcal{C}^*$ .

Problem (3.1) is equivalent to finding a  $z \in \mathcal{C}$  that satisfies the following variational inequality [7, Def. 26.19]:

$$(3.2) \quad \exists w \in F(z) \text{ s.t. } (y - z)^\top w \geq 0 \quad \forall y \in \mathcal{C}.$$

To see this first note that if we have a  $(z, w) \in F$  that satisfies (3.1), then clearly

$$y^\top w \geq z^\top w = 0$$

for all  $y \in \mathcal{C}$  since  $w \in \mathcal{C}^*$ . To see the other direction consider a  $(z, w) \in F$  with  $z \in \mathcal{C}$  that satisfies (3.2) and note that if  $z^\top w \neq 0$ , then we can take  $y = (1/2)z$  or  $y = (3/2)z$  to violate the upper bound property, so it must be the case that  $z^\top w = 0$ , then  $y^\top w \geq 0$  for all  $y \in \mathcal{C}$  implies that  $w \in \mathcal{C}^*$ .

These problems are also equivalent to the problem of finding a  $z \in \mathbb{R}^d$  that satisfies the following inclusion:

$$(3.3) \quad 0 \in F(z) + N_{\mathcal{C}}(z),$$

where  $N_{\mathcal{C}}(z)$  is the normal cone operator for cone  $\mathcal{C}$ , and is given by

$$N_{\mathcal{C}}(z) = \begin{cases} \{x \mid (y - z)^\top x \leq 0 \quad \forall y \in \mathcal{C}\}, & z \in \mathcal{C}, \\ \emptyset, & z \notin \mathcal{C}. \end{cases}$$

It is readily shown that  $N_{\mathcal{C}} = \partial I_{\mathcal{C}}$ , i.e., the subdifferential of the convex indicator function for  $\mathcal{C}$ . Therefore  $N_{\mathcal{C}}$  is maximal monotone with resolvent  $(I + N_{\mathcal{C}})^{-1}x = \Pi_{\mathcal{C}}(x)$ , the Euclidean projection onto  $\mathcal{C}$ , as can be seen using (2.3).

To see equivalence of problem (3.2) and (3.3), note that if  $z$  satisfies (3.3) then  $z \in \mathcal{C}$  and there exists  $w \in F(z)$  such that  $-w \in N_{\mathcal{C}}(z)$  and so  $z$  satisfies (3.2) and vice versa. The sum of two maximal monotone operators is also maximal monotone, so problem (3.3) is a maximal monotone inclusion problem.

An affine function  $F(z) = Mz + q$  with matrix  $M \in \mathbb{R}^{d \times d}$  and vector  $q \in \mathbb{R}^d$  is maximal monotone if and only if  $M$  is monotone, i.e.,

$$(3.4) \quad M + M^\top \succeq 0,$$

where we use the notation  $\cdot \succeq 0$  to denote membership in the positive semidefinite cone of matrices. In this case MCP( $F, \mathcal{C}$ ) is a monotone linear complementarity problem LCP( $M, q, \mathcal{C}$ ), i.e., the problem of finding  $z \in \mathbb{R}^d$  such that

$$(3.5) \quad \mathcal{C} \ni z \perp (Mz + q) \in \mathcal{C}^*.$$

When  $M$  is not monotone then the LCP may be very difficult to solve [15]. One immediate consequence of the fact that  $M$  is monotone is that

$$(3.6) \quad z^\top Mz = 0 \Leftrightarrow (M + M^\top)z = 0,$$

which can be seen from the fact that  $z^\top Mz = (1/2)z^\top(M + M^\top)z = (1/2)\|(M + M^\top)^{1/2}z\|_2^2$  for any  $z \in \mathbb{R}^d$ . We shall make use of this fact in our analysis.

**3.1. Quadratic cone programming.** As a concrete example of an LCP take the convex QCP, which is the following primal-dual problem pair:

$$(3.7) \quad \begin{array}{ll} \text{minimize} & (1/2)x^\top Px + c^\top x \\ \text{subject to} & Ax + s = b, \\ & s \in \mathcal{K}, \end{array} \quad \begin{array}{ll} \text{maximize} & -(1/2)x^\top Px - b^\top y \\ \text{subject to} & Px + A^\top y + c = 0, \\ & y \in \mathcal{K}^*, \end{array}$$

over variables  $x \in \mathbb{R}^n$ ,  $s \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^m$ , with data  $A \in \mathbb{R}^{m \times n}$ ,  $P \in \mathbb{R}^{n \times n}$ ,  $c \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$ , where  $\mathcal{K}$  is a nonempty, closed, convex cone and where  $P = P^\top \succeq 0$  (for a

derivation of the dual see [5, A.2]). When strong duality holds, the Karush–Kuhn–Tucker (KKT) conditions are necessary and sufficient for optimality [10, sect. 5.5.3]. They are given by

$$(3.8) \quad Ax + s = b, \quad Px + A^\top y + c = 0, \quad s \in \mathcal{K}, \quad y \in \mathcal{K}^*, \quad s \perp y.$$

These are primal feasibility, dual feasibility, primal and dual cone membership, and complementary slackness. The complementary slackness condition is equivalent to a zero *duality gap* condition at any optimal point, that is, for  $(x, y, s)$  that satisfy the KKT conditions we have

$$(3.9) \quad s \perp y \Leftrightarrow c^\top x + b^\top y + x^\top Px = 0.$$

The KKT conditions can be rewritten as

$$(3.10) \quad \mathbb{R}^n \times \mathcal{K}^* \ni \begin{bmatrix} x \\ y \end{bmatrix} \perp \begin{bmatrix} Px + A^\top y + c \\ b - Ax \end{bmatrix} \in \{0\}^n \times \mathcal{K},$$

which corresponds to LCP( $M, q, \mathcal{C}$ ) in variable  $z \in \mathbb{R}^d$  with

$$(3.11) \quad z = \begin{bmatrix} x \\ y \end{bmatrix}, \quad M = \begin{bmatrix} P & A^\top \\ -A & 0 \end{bmatrix}, \quad q = \begin{bmatrix} c \\ b \end{bmatrix}, \quad \mathcal{C} = \mathbb{R}^n \times \mathcal{K}^*,$$

where dimension  $d = n + m$  and  $M$  is monotone, i.e., satisfies (3.4), since  $P \succeq 0$ .

If there exists a solution to the QCP that satisfies the KKT conditions, then there exists a solution to the LCP and vice versa. On the other hand, if the QCP is strongly primal or dual infeasible, then the LCP is strongly infeasible and vice versa. In this case any  $y \in \mathbb{R}^m$  that satisfies

$$(3.12) \quad A^\top y = 0, \quad y \in \mathcal{K}^*, \quad b^\top y < 0,$$

acts a certificate that the QCP is primal infeasible (dual unbounded) [10, sect. 5.8]. Similarly, if we can find  $x \in \mathbb{R}^n$  such that

$$(3.13) \quad Px = 0, \quad -Ax \in \mathcal{K}, \quad c^\top x < 0,$$

then this is a certificate that the problem is dual infeasible (primal unbounded) [10, sect. 5.8]. We shall discuss how these certificates relate to infeasibility of LCPs in what follows.

**4. A homogeneous embedding for monotone LCPs.** As we have seen, every monotone LCP can be written as the monotone inclusion problem in (3.3). However, if the original LCP is infeasible (when there does not exist a  $z \in \mathbb{R}^d$  that satisfies the conditions (3.5)), then the monotone inclusion problem does not have a solution. In this section we derive a homogeneous embedding that always has a solution, even when the original LCP is infeasible. To do so we derive two homogeneous MCPs, one that encodes feasibility and another that encodes (strong) infeasibility. The final embedding is then an MCP involving the *union* of these two operators, which we shall show is maximal monotone.

**4.1. LCP feasibility.** Andersen and Ye developed a homogeneous embedding that encodes the feasibility conditions for MCPs [2]. When specialized to the  $d$ -dimensional LCP( $M, q, \mathcal{C}$ ) case the (single-valued) embedding operator  $\mathcal{F} : \mathbb{R}^d \times \mathbb{R}_{++} \rightarrow \mathbb{R}^{d+1}$  (where  $\mathbb{R}_{++}$  denotes the set of strictly positive real numbers) is given by

$$(4.1) \quad \mathcal{F}(z, \tau) = \begin{bmatrix} Mz + q\tau \\ -z^\top Mz/\tau - z^\top q \end{bmatrix}$$

and the embedded MCP( $\mathcal{F}, \mathcal{C}_+$ ) is to find a  $u \in \mathbb{R}^{d+1}$  such that

$$(4.2) \quad \mathcal{C}_+ \ni u \perp \mathcal{F}(u) \in \mathcal{C}_+^*,$$

where  $\mathcal{C}_+ = \mathcal{C} \times \mathbb{R}_+$ , with dual cone  $\mathcal{C}_+^* = \mathcal{C}^* \times \mathbb{R}_+$  (where  $\mathbb{R}_+$  denotes the set of nonnegative real numbers). Note that complementarity always holds, since  $u^\top \mathcal{F}(u) = 0$  for any  $u \in \mathbf{dom}(\mathcal{F})$ . Next we show that MCP( $\mathcal{F}, \mathcal{C}_+$ ) encodes the set of solutions to LCP( $M, q, \mathcal{C}$ ). If there exists a point  $z^* \in \mathbb{R}^d$  that solves LCP( $M, q, \mathcal{C}$ ), i.e., satisfies (3.5), then for any  $t > 0$

$$(4.3) \quad \mathcal{C}_+ \ni \begin{bmatrix} tz^* \\ t \end{bmatrix} \perp \begin{bmatrix} t(Mz^* + q) \\ 0 \end{bmatrix} \in \mathcal{C}_+^*$$

and so  $u = (tz^*, t) \in \mathbb{R}^d \times \mathbb{R}_{++}$  is a solution to the homogeneous embedding (4.2). Now we show the other direction. Let  $u = (z, \tau) \in \mathbf{dom}(\mathcal{F})$ , i.e.,  $\tau > 0$ , be a solution to (4.2). We know that  $z^\top (Mz + q\tau) = 0$ , and so  $(z/\tau) \perp (M(z/\tau) + q)$  and due to the positive homogeneity of cones  $z/\tau \in \mathcal{C}$  and  $(M(z/\tau) + q) \in \mathcal{C}^*$ . These imply that the point  $z/\tau$  satisfies the conditions of (3.5) and so is a solution to LCP( $M, q, \mathcal{C}$ ).

LEMMA 4.1. *The operator  $\mathcal{F}$  is monotone.*

*Proof.* Let  $u = (u_z, u_\tau) \in \mathbb{R}^d \times \mathbb{R}_{++}$ ,  $w = (w_z, w_\tau) \in \mathbb{R}^d \times \mathbb{R}_{++}$ ; then,

$$\begin{aligned} (\mathcal{F}(u) - \mathcal{F}(w))^\top (u - w) &= -(\mathcal{F}(u))^\top w - (\mathcal{F}(w))^\top u \\ &= -w_z^\top M u_z + w_\tau u_z^\top M u_z / u_\tau - u_z^\top M w_z + u_\tau w_z^\top M w_z / w_\tau \\ &= u_\tau w_\tau (M u_z / u_\tau - M w_z / w_\tau)^\top (u_z / u_\tau - w_z / w_\tau) \\ &\geq 0 \end{aligned}$$

since  $M$  is monotone and  $u_\tau w_\tau > 0$ . □

Although  $\mathcal{F}$  is monotone, it is not *maximal* monotone, which is a required property for DR splitting to have guaranteed convergence. In order to extend the operator to be maximal we must consider infeasibility of the original LCP, which we do next.

**4.2. LCP infeasibility.** Let us denote by  $\mathcal{A} = \{(z, w) \mid w = -(Mz + q)\}$ . LCP( $M, q, \mathcal{C}$ ) is feasible if and only if there exists a point  $(z, w) \in N_{\mathcal{C}} \cap \mathcal{A}$ . To see this observe that any such point satisfies  $-(Mz + q) = w \in N_{\mathcal{C}}(z)$ , so  $0 \in (Mz + q) + N_{\mathcal{C}}(z)$ , i.e.,  $z$  satisfies (3.3). If  $N_{\mathcal{C}} \cap \mathcal{A} = \emptyset$ , then no such point exists and the problem is *infeasible*. A stronger condition is that the distance between the sets  $N_{\mathcal{C}}$  and  $\mathcal{A}$  is strictly positive, that is,

$$\mathbf{dist}(N_{\mathcal{C}}, \mathcal{A}) = \inf_{(z_1, w_1) \in N_{\mathcal{C}}, (z_2, w_2) \in \mathcal{A}} \|(z_1, w_1) - (z_2, w_2)\| > 0,$$

in which case we refer to the problem as *strongly infeasible* [42, 40, 41]. A necessary and sufficient condition for this is that the sets are *strongly separated* [63, Chap. 11], which is the existence of a strongly separating hyperplane with normal vector  $(\mu, \lambda) \in \mathbb{R}^d \times \mathbb{R}^d$  that satisfies

$$\inf_{(z, w) \in \mathcal{A}} (z^\top \mu + w^\top \lambda) > 0, \quad \sup_{(z, w) \in N_{\mathcal{C}}} (z^\top \mu + w^\top \lambda) \leq 0,$$

since  $N_{\mathcal{C}}$  is a cone [63, Thm. 11.7]. We can simplify this by substituting  $w = -(Mz+q)$  into the first condition, yielding

$$\inf_{z \in \mathbb{R}^d} (z^\top (\mu - M^\top \lambda) - \lambda^\top q) > 0,$$

which implies that  $\mu = M^\top \lambda$ , and consequently that  $\lambda^\top q < 0$ . This brings us to necessary and sufficient conditions for strong infeasibility of  $\text{LCP}(M, q, \mathcal{C})$ , which is the existence of a  $\lambda \in \mathbb{R}^d$  such that

$$(4.4) \quad \lambda^\top q < 0, \quad \sup_{(z,w) \in N_{\mathcal{C}}} \lambda^\top (Mz + w) \leq 0.$$

Next we establish that the above conditions on  $\lambda$  can be embedded into another LCP.

LEMMA 4.2. *LCP( $M, q, \mathcal{C}$ ) is strongly infeasible if and only if there exists a  $\lambda \in \mathbb{R}^d$  with  $\lambda^\top q < 0$  that solves LCP( $M, 0, \mathcal{C}$ ), i.e.,*

$$(4.5) \quad \mathcal{C} \ni \lambda \perp M\lambda \in \mathcal{C}^*.$$

*Proof.* First, we show that any certificate of strong infeasibility solves (4.5). Consider the second condition in (4.4); setting  $z = 0$  yields  $w^\top \lambda \leq 0$  for all  $w \in N_{\mathcal{C}}(0) = -\mathcal{C}^*$ , and so  $\lambda \in \mathcal{C}$ . For any  $z \in \mathcal{C}$  we know that  $0 \in N_{\mathcal{C}}(z)$  and so  $\lambda^\top Mz \leq 0$ , which implies that  $-M^\top \lambda \in \mathcal{C}^*$ . Together these tell us that  $\lambda^\top M\lambda \leq 0$ , but since  $M$  is monotone it must be that  $\lambda^\top M\lambda = 0$  and therefore  $M\lambda = -M^\top \lambda$ , using (3.6). Putting it together with the fact that  $\lambda^\top q < 0$  yields the final result.

Now we show the other direction: assume  $\lambda \in \mathbb{R}^d$  satisfies (4.5) with  $\lambda^\top q < 0$ . We must show that this satisfies the second condition in (4.4). Take any  $(z, w) \in N_{\mathcal{C}}$  and  $x \in \mathcal{C}$ ; then from the definition of normal cones  $x^\top w \leq z^\top w$ . If  $x^\top w > 0$ , then there must exist some  $t > 0$  such that  $tx^\top w > z^\top w$ , and since  $tx \in \mathcal{C}$  this would contradict that fact that  $w \in N_{\mathcal{C}}(z)$ . So it must be the case that  $x^\top w \leq 0$ . Since  $x$  was arbitrary in  $\mathcal{C}$  it implies that  $-w \in \mathcal{C}^*$ , and so  $\lambda^\top w \leq 0$  due to  $\lambda \in \mathcal{C}$ . Since  $\lambda^\top M\lambda = 0$  we know that  $M\lambda = -M^\top \lambda \in \mathcal{C}^*$  from (3.6), so  $z^\top (M^\top \lambda) \leq 0$ . Summing these two yields  $\lambda^\top (Mz + w) \leq 0$  for any  $(z, w) \in N_{\mathcal{C}}$ .  $\square$

We call any  $\lambda$  that satisfies (4.4) a proof or certificate of (strong) infeasibility. The existence of such a  $\lambda$  precludes the existence of  $(z, w) \in N_{\mathcal{C}} \cap \mathcal{A}$ , and any  $(z, w) \in N_{\mathcal{C}} \cap \mathcal{A}$  acts as a certificate that there is no  $\lambda$  satisfying (4.4). In other words at most one of (4.4) and (3.5) has a solution and they are therefore *weak alternatives*. This can also be proven directly from the LCPs: Assume that we have found both a  $z \in \mathbb{R}^d$  that solves  $\text{LCP}(M, q, \mathcal{C})$  and a  $\lambda \in \mathbb{R}^d$  that solves  $\text{LCP}(M, 0, \mathcal{C})$  with  $\lambda^\top q < 0$ . Then  $z + \lambda \in \mathcal{C}$  and  $M(z + \lambda) + q \in \mathcal{C}^*$  and from cone duality  $0 \leq (z + \lambda)^\top (M(z + \lambda) + q) = \lambda^\top q < 0$ , which is a contradiction.

In the special case of a QCP satisfying strong duality then *exactly* one of those two systems has a solution and they are *strong alternatives* [10, sect. 5.8].

**4.2.1. QCP infeasibility.** Here we show that the conditions in (4.5) are exactly equivalent to the conditions of (strong) primal infeasibility (3.12) or (strong) dual infeasibility (3.13) in the case where we are solving a QCP, and that any certificate for one can be converted into a certificate for the other.

First, consider the case where  $y \in \mathbb{R}^m$  is a certificate of primal infeasibility for the QCP; then  $\lambda = (0, y) \in \mathbb{R}^n \times \mathbb{R}^m$  is a certificate for the LCP since it is readily verified to satisfy the conditions in (4.5) with  $\lambda^\top q = b^\top y < 0$ . Similarly, if  $x \in \mathbb{R}^n$  is a



certificate of dual infeasibility for the QCP, then  $\lambda = (x, 0) \in \mathbb{R}^n \times \mathbb{R}^m$  is a certificate of infeasibility for the LCP by the same logic.

Now consider  $\lambda = (x, y) \in \mathbb{R}^n \times \mathbb{R}^m$  a certificate of infeasibility for  $\text{LCP}(M, q, \mathcal{C})$  corresponding to a QCP, in which case using (4.5) we have

$$(4.6) \quad \mathbb{R}^n \times \mathcal{K}^* \ni \begin{bmatrix} x \\ y \end{bmatrix} \perp \begin{bmatrix} Px + A^\top y \\ -Ax \end{bmatrix} \in \{0\}^n \times \mathcal{K}.$$

First note that  $y \in \mathcal{K}^*$  and  $-Ax \in \mathcal{K}$ . The second orthogonality condition implies that  $y^\top Ax = 0$ . From this and the first orthogonality condition we can infer that  $x^\top Px = 0$  and so  $Px = 0$ , and therefore  $A^\top y = 0$  due to the  $\{0\}^n$  cone membership. Finally,  $q^\top \lambda = c^\top x + b^\top y < 0$  by assumption, and so at least one of  $c^\top x$  or  $b^\top y$  is negative. If  $c^\top x < 0$ , then  $x$  is a certificate for dual infeasibility for the QCP since it satisfies (3.13); on the other hand if  $b^\top y < 0$ , then  $y$  is a certificate of primal infeasibility since it satisfies (3.12). If both  $c^\top x$  and  $b^\top y$  are negative, then the original problem is both primal and dual infeasible.

**4.3. Infeasibility embedding.** Here we introduce a homogeneous operator that encodes the infeasibility conditions for  $\text{LCP}(M, q, \mathcal{C})$ . It will become clear why we need this operator in the next section when we use it to derive the complete embedding. Based on Lemma 4.2 we define the operator  $\mathcal{I}$  on  $\mathbb{R}^{d+1}$  as

$$(4.7) \quad \mathcal{I}(z, \tau) = \left\{ \begin{bmatrix} Mz \\ \kappa \end{bmatrix} \mid \kappa \leq -z^\top q \right\}, \quad \mathbf{dom}(\mathcal{I}) = \{(z, 0) \mid z^\top Mz = 0\},$$

where  $(z, \tau) \in \mathbb{R}^d \times \mathbb{R}$ . Consider  $\text{MCP}(\mathcal{I}, \mathcal{C}_+)$ , that is, the problem of finding  $u \in \mathbb{R}^{d+1}$  for which

$$(4.8) \quad \exists v \in \mathcal{I}(u) \text{ s.t. } \mathcal{C}_+ \ni u \perp v \in \mathcal{C}_+^*.$$

Note that again complementarity is always satisfied, i.e.,  $u^\top v = 0$  for all  $(u, v) \in \mathcal{I}$ . If  $\lambda$  is a certificate of infeasibility for  $\text{LCP}(M, q, \mathcal{C})$ , then  $(\lambda, 0) \in \mathbf{dom}(\mathcal{I})$  and

$$\begin{bmatrix} M\lambda \\ -\lambda^\top q \end{bmatrix} \in \mathcal{I}(\lambda, 0),$$

and therefore  $(\lambda, 0)$  is a solution to  $\text{MCP}(\mathcal{I}, \mathcal{C}_+)$ . On the other hand, any solution  $u$  to  $\text{MCP}(\mathcal{I}, \mathcal{C}_+)$  such that  $(w, \kappa) = v \in \mathcal{I}(u)$  with  $\kappa > 0$  yields a certificate of infeasibility for  $\text{LCP}(M, q, \mathcal{C})$ .

**LEMMA 4.3.** *The operator  $\mathcal{I}$  is monotone.*

*Proof.* Let  $u = (u_z, 0) \in \mathbb{R}^d \times \mathbb{R}$  and  $w = (w_z, 0) \in \mathbb{R}^d \times \mathbb{R}$  such that  $u, w \in \mathbf{dom}(\mathcal{I})$ , then,

$$\begin{aligned} (\mathcal{I}(u) - \mathcal{I}(w))^\top (u - w) &= -\mathcal{I}(u)^\top w - \mathcal{I}(w)^\top u \\ &= -u_z^\top M w_z - w_z^\top M u_z \\ &= -w_z^\top (M + M^\top) u_z \\ &= 0 \end{aligned}$$

since  $(M + M^\top)u_z = (M + M^\top)w_z = 0$  using (3.6). □

**4.4. Final embedding.** We have two homogeneous monotone operators,  $\mathcal{F}$  and  $\mathcal{I}$ , with associated problems  $\text{MCP}(\mathcal{F}, \mathcal{C}_+)$  and  $\text{MCP}(\mathcal{I}, \mathcal{C}_+)$  that encode feasibility and infeasibility of the original problem  $\text{LCP}(M, q, \mathcal{C})$ , respectively. However, neither of these operators is maximal. In the next section we show that the *union* of the two operators is maximal monotone, and the associated MCP encodes both feasibility and infeasibility of the original LCP. Let

$$\mathcal{Q} = \mathcal{F} \cup \mathcal{I}$$

with  $\mathbf{dom}(\mathcal{Q}) = \mathbf{dom}(\mathcal{F}) \cup \mathbf{dom}(\mathcal{I})$ . The operator  $\mathcal{Q}$  satisfies complementarity, i.e.,  $u^\top v = 0$  for all  $(u, v) \in \mathcal{Q}$ , and is positively homogeneous, i.e.,  $\mathcal{Q}(tu) = t\mathcal{Q}(u)$  for any  $t > 0$ . The embedded problem is to solve  $\text{MCP}(\mathcal{Q}, \mathcal{C}_+)$ , i.e., find a  $u \in \mathbb{R}^{d+1}$  for which

$$(4.9) \quad \exists v \in \mathcal{Q}(u) \text{ s.t. } \mathcal{C}_+ \ni u \perp v \in \mathcal{C}_+^*,$$

which from section 3 we know is equivalent to the monotone inclusion

$$(4.10) \quad 0 \in \mathcal{Q}(u) + N_{\mathcal{C}_+}(u).$$

As we shall show, both  $\mathcal{Q}$  and  $N_{\mathcal{C}_+}$  are maximal monotone and so we can apply operator splitting methods to solve this problem. First, we discuss how the solutions to  $\text{MCP}(\mathcal{Q}, \mathcal{C}_+)$  encode the solutions or certificates of infeasibility to  $\text{LCP}(M, q, \mathcal{C})$ . Let  $u^* = (z^*, \tau^*) \in \mathbb{R}^d \times \mathbb{R}$  be any point that satisfies (4.9), and let  $(w^*, \kappa^*) = v^* \in \mathcal{Q}(u^*)$ . From complementarity we know that

$$(u^*)^\top v^* = (z^*)^\top w^* + \tau^* \kappa^* = 0.$$

However,  $(z^*)^\top w^* \geq 0$  and  $\tau^* \kappa^* \geq 0$  since  $\mathcal{C}_+$  and  $\mathcal{C}_+^*$  are dual, and so it must be that  $z^* \perp w^*$  and at most one of  $\tau^*$  and  $\kappa^*$  can be positive. When  $\tau^* > 0$  then  $\kappa^* = 0$ ,  $u^* \in \mathbf{dom}(\mathcal{F})$ ,  $v^* = \mathcal{F}(u^*)$ , the problem is feasible and a solution to  $\text{LCP}(M, q, \mathcal{C})$  can be derived from  $u^*$ . When  $\kappa^* > 0$  then  $\tau^* = 0$ ,  $u^* \in \mathbf{dom}(\mathcal{I})$ ,  $v^* \in \mathcal{I}(u^*)$ , the problem is infeasible and a certificate of infeasibility of  $\text{LCP}(M, q, \mathcal{C})$  can be obtained from  $u^*$ . The next case to consider is when  $\tau^* = \kappa^* = 0$  with  $u \neq 0$ . This is pathological and rarely arises in practice [76]. We can rule out some situations for this case; for example, if the set of solutions to the LCP is nonempty and bounded, then this pathology cannot occur. On the other hand, if the LCP is weakly infeasible, then the only solutions to the homogeneous embedding have this form. This includes, for example, feasible QCPs that do not satisfy strong duality. However, in that case it may be possible to modify the problem using facial reduction techniques [62] or to understand the pathology by examining how the iterates generated by DR splitting behave [38].

These cases are summarized in Table 1. The only other possibility we must consider is the trivial solution  $u = 0$ , which is always a solution to  $\text{MCP}(\mathcal{Q}, \mathcal{C}_+)$ , no matter the problem data. However, we shall prove later that DR splitting will not converge to zero if properly initialized, so we can safely ignore this possibility.

TABLE 1  
*How the solutions of the MCP relate to the status of the LCP.*

	$\tau^* > 0$	$\tau^* = 0$
$\kappa^* > 0$	N/A	Infeasible
$\kappa^* = 0$	Solved	Pathological

**4.5. Maximal monotonicity of  $\mathcal{Q}$ .** In order to apply DR splitting to problem (4.10) we need  $\mathcal{Q}$  to be *maximal monotone*, without which convergence is not guaranteed.

LEMMA 4.4. *The operator  $\mathcal{Q} = \mathcal{F} \cup \mathcal{I}$  is maximal monotone.*

*Proof.* Since  $\mathcal{F}$  and  $\mathcal{I}$  are both monotone, to show that  $\mathcal{Q}$  is monotone we need only consider points  $u \in \mathbf{dom}(\mathcal{F})$  and  $w \in \mathbf{dom}(\mathcal{I})$ . Let  $u = (u_z, u_\tau) \in \mathbb{R}^d \times \mathbb{R}_{++}$ ,  $w = (w_z, 0) \in \mathbb{R}^d \times \mathbb{R}$ , and  $(Mw_z, \kappa) \in \mathcal{I}(w)$ ; then

$$\begin{aligned} (\mathcal{Q}(u) - \mathcal{Q}(w))^\top (u - w) &= -\mathcal{Q}(u)^\top w - \mathcal{Q}(w)^\top u \\ &= -\mathcal{F}(u)^\top w - \mathcal{I}(w)^\top u \\ &\ni -w_z^\top (Mu_z + qu_\tau) - u_z^\top (Mw_z) - u_\tau \kappa \\ &= -u_z^\top (M + M^\top)w_z - u_\tau (\kappa + w_z^\top q) \\ &\geq 0 \end{aligned}$$

since  $(M + M^\top)w_z = 0$  and  $\kappa \leq -w_z^\top q$ . Since it holds for any  $\kappa$  satisfying the inclusion this establishes that  $\mathcal{Q}$  is monotone; next we shall show maximality.

For any monotone operator there exists a maximal monotone extension of it with domain contained in the closure of the convex hull of its domain [7, Thm. 21.9]. The domain of  $\mathcal{F}$  is  $\mathbb{R}^d \times \mathbb{R}_{++}$ , which is convex, and so there exists a maximal monotone extension of  $\mathcal{F}$  with domain contained in  $\mathbb{R}^d \times \mathbb{R}_+$ . Let  $\bar{\mathcal{F}}$  denote such an extension. We shall show that  $\bar{\mathcal{F}}$  is unique and  $\bar{\mathcal{F}} = \mathcal{Q}$ .

To construct the extension we need to find all pairs  $(p, r)$  such that  $\mathcal{F} \cup \{p, r\}$  is monotone, with  $p \in \mathbb{R}^d \times \mathbb{R}_+$ . Since  $\mathcal{F}$  is continuous on the interior of its domain we can use standard arguments to show that no such extension pair with  $p \in \mathbf{dom}(\mathcal{F})$  exists [7]. So any extension pairs  $(p, r)$  must have  $p$  on the boundary of  $\mathbb{R}^d \times \mathbb{R}_+$ , which, if we let  $p = (p_z, p_\tau) \in \mathbb{R}^d \times \mathbb{R}$ , corresponds to points with  $p_\tau = 0$ . Let  $u = (z, \tau) \in \mathbf{dom}(\mathcal{F})$  and consider points  $p = (p_z, 0) \in \mathbb{R}^d \times \mathbb{R}$  and  $r = (r_z, r_\tau) \in \mathbb{R}^d \times \mathbb{R}$ . The monotone property implies that  $(p, r)$  must satisfy

$$\begin{aligned} 0 &\leq (\mathcal{F}(u) - r)^\top (u - p) \\ &= -\mathcal{F}(u)^\top p - r^\top (u - p) \\ &= -p_z^\top Mz - \tau p_z^\top q - r_z^\top (z - p_z) - r_\tau \tau. \end{aligned}$$

Since  $z$  is arbitrary this implies that  $M^\top p_z + r_z = 0$ , which in turn implies that

$$0 \leq -\tau(p_z^\top q + r_\tau) - p_z^\top Mp_z.$$

Letting  $\tau \rightarrow 0$  we get  $p_z^\top Mp_z \leq 0$ , but since  $M$  is monotone this implies that

$$(4.11) \quad p_z^\top Mp_z = 0$$

and so  $Mp_z = -M^\top p_z$  from (3.6), which yields

$$(4.12) \quad Mp_z = r_z.$$

Finally, since  $\tau \geq 0$  we have

$$(4.13) \quad r_\tau \leq -p_z^\top q.$$

The conditions (4.11), (4.12), (4.13) on  $(p, r)$  are exactly the conditions for  $(p, r) \in \mathcal{I}$  from the definition of  $\mathcal{I}$  in (4.7). Thus all extension pairs must be elements of  $\mathcal{I}$  and so  $\overline{\mathcal{F}} \subseteq \mathcal{F} \cup \mathcal{I} = \mathcal{Q}$ . However, it cannot be the case that  $\overline{\mathcal{F}} \subset \mathcal{Q}$  strictly, as  $\mathcal{Q}$  is monotone this would violate the maximality of  $\overline{\mathcal{F}}$ . Therefore we can conclude that  $\overline{\mathcal{F}} = \mathcal{Q}$ , i.e.,  $\mathcal{Q} = \mathcal{F} \cup \mathcal{I}$  is a maximal monotone extension of  $\mathcal{F}$ .  $\square$

**5. Douglas–Rachford splitting for LCPs.** We have discussed how the feasibility and infeasibility conditions for an LCP can be embedded into a single homogeneous MCP. In this section we apply DR splitting to  $\text{MCP}(\mathcal{Q}, \mathcal{C}_+)$ ; the algorithm that solves the homogeneous embedded problem is the main result of this manuscript.

We have established that the operator  $\mathcal{Q}$  is maximal monotone (as is  $N_{\mathcal{C}_+}$ ). This implies that DR splitting applied to  $\text{MCP}(\mathcal{Q}, \mathcal{C}_+)$  will enjoy the convergence properties discussed in section 2. That is, from any initial  $w^0 \in \mathbb{R}^{d+1}$  the procedure for  $k = 0, 1, \dots$ ,

$$(5.1) \quad \begin{aligned} \tilde{u}^{k+1} &= (I + \mathcal{Q})^{-1}w^k, \\ u^{k+1} &= \Pi_{\mathcal{C}_+}(2\tilde{u}^{k+1} - w^k), \\ w^{k+1} &= w^k + u^{k+1} - \tilde{u}^{k+1}, \end{aligned}$$

will converge to a fixed point from which we can derive a solution or a certificate of infeasibility for the original LCP( $M, q, \mathcal{C}$ ). The remaining difficulty is the evaluation of the resolvent of  $\mathcal{Q}$ , which we discuss in what follows.

By way of comparison, we can also apply DR splitting to LCP( $M, q, \mathcal{C}$ ) directly, which yields the following procedure: from any initial  $w^0 \in \mathbb{R}^d$  for  $k = 0, 1, \dots$ ,

$$(5.2) \quad \begin{aligned} \tilde{u}^{k+1} &= (I + M)^{-1}(w^k - q), \\ u^{k+1} &= \Pi_{\mathcal{C}}(2\tilde{u}^{k+1} - w^k), \\ w^{k+1} &= w^k + u^{k+1} - \tilde{u}^{k+1}. \end{aligned}$$

If a solution to LCP( $M, q, \mathcal{C}$ ) exists, then this procedure will converge; otherwise it has no fixed point and will not converge.

**5.1. Evaluating the resolvent of  $\mathcal{Q}$ .** Since  $\mathcal{Q}$  is maximal monotone we know that the resolvent is single-valued and has full domain [65]. At time-step  $k$  of DR splitting we must solve a system of equations involving the resolvent of  $\mathcal{Q}$ , that is, solve

$$\begin{bmatrix} z \\ \tau \end{bmatrix} = (I + \mathcal{Q})^{-1} \begin{bmatrix} \mu^k \\ \eta^k \end{bmatrix}$$

for a fixed right-hand side  $(\mu^k, \eta^k) \in \mathbb{R}^d \times \mathbb{R}$ . Suppose for a moment we know that  $(z, \tau) \in \mathbf{dom}(\mathcal{F})$ , i.e.,  $\tau > 0$ ; then using (4.1) we must solve

$$(5.3) \quad \begin{aligned} (I + M)z + q\tau &= \mu^k, \\ \tau^2 - \tau(\eta^k + z^\top q) - z^\top Mz &= 0 \end{aligned}$$

for  $z \in \mathbb{R}^d$  and  $\tau > 0$ . Since  $M$  is monotone we have  $z^\top Mz \geq 0$ , and so one root of the quadratic equation is nonnegative and one is nonpositive, and since  $(z, \tau) \in \mathbf{dom} \mathcal{Q}$  it is the nonnegative root that corresponds to the solution. The solution to these equations also encodes the solution when  $(z, \tau) \in \mathbf{dom}(\mathcal{I})$ , since if  $z^\top Mz = 0$ , then the nonnegative root is given by  $\tau = \max(0, \eta^k + z^\top q)$ . In other words,  $\tau = 0$  if and only if  $\eta^k \leq -z^\top q$  and  $z^\top Mz = 0$ , which are the conditions for  $(z, \tau) \in \mathbf{dom}(\mathcal{I})$  in

(4.7). This means the solution of (5.3) for  $\tau \geq 0$  yields the resolvent of  $\mathcal{Q}$  for any right-hand side. Let us denote by

$$p^k = (I + M)^{-1}\mu^k, \quad r = (I + M)^{-1}q,$$

then we have

$$z = p^k - r\tau$$

for unknown  $\tau \geq 0$ , and note that since  $r$  is constant for all iterations we only need to compute it once at the start of the procedure and then reuse this cached value thereafter. To solve for  $\tau$  we substitute  $z = p^k - r\tau$  into the quadratic equation (5.3) yielding

$$\begin{aligned} 0 &= \tau^2 - \tau(\eta^k + z^\top q) - z^\top((I + M)z - z) \\ (5.4) \quad &= \tau^2 - \tau(\eta^k + (p^k - r\tau)^\top q) - (p^k - r\tau)^\top(\mu^k - q\tau - p^k + r\tau) \\ &= \tau^2(1 + r^\top r) + \tau(r^\top \mu^k - 2r^\top p^k - \eta^k) + (p^k)^\top(p^k - \mu^k), \end{aligned}$$

and for brevity we denote  $\text{root}_+(\mu^k, \eta^k, p^k, r)$  to be the nonnegative root of the quadratic equation (5.4) when evaluated with input values  $(\mu^k, \eta^k, p^k, r)$ . Specifically, let  $a = 1 + r^\top r$ ,  $b^k = r^\top \mu^k - 2r^\top p^k - \eta^k$ , and  $c^k = (p^k)^\top(p^k - \mu^k)$ ; then

$$(5.5) \quad \text{root}_+(\mu^k, \eta^k, p^k, r) = \left(-b^k + \sqrt{(b^k)^2 - 4ac^k}\right) / 2a.$$

Since the resolvent has full domain it always has a real-valued solution for any input, which implies that the above quadratic equation always has real roots. This fact can also be seen directly from the equations by noting that  $(b^k)^2 \geq 0$ ,  $a = 1 + r^\top r \geq 0$ , and  $c^k = (p^k)^\top(p^k - \mu^k) = -(p^k)^\top Mp^k \leq 0$  since  $M$  is monotone, and so  $(b^k)^2 - 4ac^k \geq 0$ .

**5.2. Final algorithm.** With the resolvent of  $\mathcal{Q}$  in place we are ready to present DR splitting applied to problem (4.10) as Algorithm 5.1. The  $u^k, \tilde{u}^k, w^k$  terms in Algorithm 5.1 are simply to relate the procedure to that described in (5.1).

Note that neither Algorithm 5.1 nor the procedure described in (5.2) has any explicit hyperparameters (e.g., step-size), though in practice the relative scaling of the problem data can have a large impact on the convergence of the algorithm and most practical solvers based on DR splitting or ADMM implement some sort of heuristic data rescaling [56, 24, 14, 28].

Algorithm 5.1 and the procedure in (5.2) differ only in that Algorithm 5.1 maintains an additional set of scalar parameters ( $\tau, \tilde{\tau}$ , and  $\eta$ ), and consequently the computational costs of the two algorithms are essentially the same. However, (5.2) will not converge if the LCP is infeasible, whereas Algorithm 5.1 will always converge and will produce a certificate of infeasibility should one exist. In fact, the procedure in (5.2) can be interpreted as Algorithm 5.1 where we fix the scalar parameters  $\tau = \tilde{\tau} = \eta = 1$ . It may be the case that this is not the best choice for any particular problem and allowing these scale parameters to vary makes the problem easier, even for feasible cases. We shall present some preliminary evidence of this effect in the numerical experiments sections.

For the special case of QCPs with  $P = 0$  the problem reduces to a linear cone program of the form that the original SCS algorithm [56] was developed to tackle. Unsurprisingly, we recover SCS from Algorithm 5.1 in this case (modulo the change of variables required to go from ADMM to DR splitting), with the minor difference that Algorithm 5.1 constrains the  $\tilde{\tau}^k$  variable to always be nonnegative, which is not the case in SCS.

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**Algorithm 5.1** DR splitting for the homogeneous embedding of LCPs
 

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**Input:**  $\text{LCP}(M, q, \mathcal{C})$   
 compute  $r = (I + M)^{-1}q$   
 initialize  $\mu^0 \in \mathbb{R}^d, \eta^0 > 0$   
**for**  $k = 0, 1, \dots$  **do**

$$\tilde{u}^{k+1} : \begin{cases} p^k &= (I + M)^{-1}\mu^k \\ \tilde{\tau}^{k+1} &= \text{root}_+(\mu^k, \eta^k, p^k, r) \\ \tilde{z}^{k+1} &= p^k - r\tilde{\tau}^{k+1} \end{cases}$$

$$u^{k+1} : \begin{cases} z^{k+1} &= \Pi_{\mathcal{C}}(2\tilde{z}^{k+1} - \mu^k) \\ \tau^{k+1} &= \Pi_{\mathbb{R}_+}(2\tilde{\tau}^{k+1} - \eta^k) \end{cases}$$

$$w^{k+1} : \begin{cases} \mu^{k+1} &= \mu^k + z^{k+1} - \tilde{z}^{k+1} \\ \eta^{k+1} &= \eta^k + \tau^{k+1} - \tilde{\tau}^{k+1} \end{cases}$$

**end for**

---

**5.3. Eliminating the trivial solution.** Since problem (4.10) is homogeneous the point  $u = 0$  is a solution no matter the data, and we might worry that our approach will converge to zero, or to a point so close to zero that it is impossible to recover a solution to the original LCP in a numerically stable way. Here we generalize a result from [56] to show that this cannot happen so long as the procedure is initialized correctly.

LEMMA 5.1. *Fix  $w^0 \in \mathbb{R}^d$  and consider the sequence  $w^{k+1} = \mathcal{T}(w^k)$  for  $k = 0, 1, \dots$ , generated by  $\mathcal{T} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . If*

1.  $\mathcal{T}$  is positively homogeneous, i.e.,  $\mathcal{T}(tv) = t\mathcal{T}(v)$  for any  $t > 0, v \in \mathbb{R}^d$ ,
2.  $\mathcal{T}$  has a nonzero fixed point  $w^* \in \mathbb{R}^d$  which satisfies  $(w^*)^\top w^0 > 0$ ,
3.  $\mathcal{T}$  is nonexpansive toward any fixed point, i.e.,  $\|\mathcal{T}(v) - w^*\|_2 \leq \|v - w^*\|_2$  for any  $v \in \mathbb{R}^d$ ,

then for all  $k$ ,

$$\|w^k\|_2 \geq \frac{(w^*)^\top w^0}{\|w^*\|_2} > 0.$$

*Proof.* Since  $\mathcal{T}$  is positively homogeneous the point  $tw^*$  is also a fixed point for any  $t > 0$ , and since  $\mathcal{T}$  is nonexpansive toward any fixed point we have

$$\begin{aligned} \|w^k - tw^*\|_2^2 &\leq \|w^0 - tw^*\|_2^2 \\ \Rightarrow -2t(w^*)^\top w^k &\leq \|w^0\|_2^2 - 2t(w^*)^\top w^0 \\ \Rightarrow \|w^*\|_2 \|w^k\|_2 &\geq (w^*)^\top w^0 - \|w^0\|_2^2 / 2t, \end{aligned}$$

where we used Cauchy–Schwarz in the last line, and letting  $t \rightarrow \infty$  yields the desired result.  $\square$

If the operator  $\mathcal{T}$  corresponds to one step of DR splitting, then it is *globally* non-expansive [7]. When applied to  $\text{MCP}(\mathcal{Q}, \mathcal{C}_+)$  DR splitting is positively homogeneous, since both  $\mathcal{Q}$  and  $N_{\mathcal{C}}$  are positively homogeneous. Finally, if we assume that either

an optimal solution or a certificate of infeasibility exists for  $\text{LCP}(M, q, \mathcal{C})$ , then it has a nonzero fixed point, and since  $w^* \in u^* + \mathcal{Q}(u^*)$  [7], where  $u^*$  is a solution to  $\text{MCP}(\mathcal{Q}, \mathcal{C}_+)$ , it is easy to initialize in such a way that the condition  $(w^*)^\top w^0 > 0$  is satisfied. For example, we can set the last entry of  $w^0$  to one and the rest of the entries to zero. Therefore, under normal conditions DR splitting satisfies the conditions of the lemma and so Algorithm 5.1 will converge to a point that is bounded away from zero.

**5.4. Convergence of Algorithm 5.1.** The convergence guarantees for DR splitting tell us that  $u^k \rightarrow u^*$ ,  $w^k \rightarrow w^* \in u^* + \mathcal{Q}(u^*)$ , and  $\|u^k - \tilde{u}^k\| \rightarrow 0$ , where  $u^*$  is a solution to  $\text{MCP}(\mathcal{Q}, \mathcal{C}_+)$  [7, Thm. 26.11]. A solution always exists since  $u^* = 0$  is a solution, though we know from Lemma 5.1 that the procedure will not converge to zero under benign conditions.

Consider the sequence defined as  $v^{k+1} = u^{k+1} + w^k - 2\tilde{u}^{k+1}$  for  $k = 0, 1, \dots$ . This sequence converges to  $\mathcal{Q}(u^*)$  since

$$v^{k+1} = u^{k+1} + w^k - 2\tilde{u}^{k+1} \rightarrow w^* - u^* \in \mathcal{Q}(u^*).$$

Furthermore, substituting in for  $u^{k+1}$  from (5.1) combined with the Moreau decomposition [60, 58] yields

$$\begin{aligned} v^{k+1} &= u^{k+1} + w^k - 2\tilde{u}^{k+1} \\ &= \Pi_{\mathcal{C}_+}(2\tilde{u}^{k+1} - w^k) + w^k - 2\tilde{u}^{k+1} \\ &= \Pi_{\mathcal{C}_+^*}(-2\tilde{u}^{k+1} + w^k). \end{aligned}$$

That is,  $u^{k+1}$  and  $v^{k+1}$  correspond to the orthogonal Moreau decomposition of  $2\tilde{u}^{k+1} - w^k$  onto the cone  $\mathcal{C}_+$  and its polar (negative dual) cone, which implies that  $v^k \in \mathcal{C}_+^*$  and  $u^k \perp v^k$  for all  $k$ . In summary, the iterates  $(u^k, v^k)$  satisfy

$$(5.6) \quad \mathcal{C}_+ \ni u^k \perp v^k \in \mathcal{C}_+^*$$

for all  $k$ , and the condition  $v^k \in \mathcal{Q}(u^k)$  holds in the limit, i.e., the pair  $(u^k, v^k)$  eventually satisfies the conditions in (4.9). Now take the special case of a QCP where  $u^k = (x^k, y^k, \tau^k) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$  and  $v^k = (0, s^k, \kappa^k) \in \{0\}^n \times \mathbb{R}^m \times \mathbb{R}$ . If  $\tau^k \rightarrow \tau^* > 0$ , then, since  $v^k$  converges to  $\mathcal{Q}(u^k)$ , these iterates will in the limit provide a solution which satisfies the KKT conditions (3.10), i.e.,  $(x^k/\tau^k, s^k/\tau^k, y^k/\tau^k) \rightarrow (x^*, s^*, y^*)$ . Due to (5.6) we know that  $s^k/\tau^k \in \mathcal{K}$ ,  $y^k/\tau^k \in \mathcal{K}^*$ , and  $s^k/\tau^k \perp y^k/\tau^k$  for all  $k$  so three of the KKT conditions are always satisfied by this sequence. Therefore to check for optimality we only need to test that the primal residual, dual residual, and duality gap defined in (3.9) are less than some tolerance. On the other hand if  $\kappa^k \rightarrow \kappa^* > 0$ , then the iterates will converge to a certificate of primal infeasibility (3.12) or dual infeasibility (3.13). To check for infeasibility we only need to check that the certificate residuals are below some tolerance and that either  $c^\top x^k < 0$  or  $b^\top y^k < 0$ , since both  $s^k$  and  $y^k$  satisfy the cone membership requirement.

**6. Implementation details for QCPs.** The algorithm we have derived applies to any monotone LCP. In this section we discuss how to perform the steps in Algorithm 5.1 efficiently for the QCP special case.

**6.1. Solving the linear system.** In both the procedure described in (5.2) and Algorithm 5.1 we need to solve a system of equations with the same matrix at every iteration. For the specific case of a QCP the linear system can be written

$$\begin{bmatrix} I + P & A^T \\ A & -I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mu_x \\ -\mu_y \end{bmatrix}$$

for  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$  and right-hand side  $(\mu_x, \mu_y) \in \mathbb{R}^n \times \mathbb{R}^m$ . There are two main ways we consider to solve this system of equations. The first way is a *direct* method, which solves the system exactly by initially computing a sparse permuted  $LDL^\top$  factorization of the matrix [17], caching this factorization, and reusing it every iteration thereafter. In the majority of cases the factorization cost is greater than the solve cost using the factors, so once the initial work is done the subsequent iterations are much cheaper. Since  $P \succeq 0$  this matrix above is *quasidefinite*, which implies that the  $LDL^\top$  factorization exists for any symmetric permutation [74].

Alternatively, we can apply an *indirect* method to solve the system approximately at each iteration. DR splitting is robust to inexact evaluations of the resolvent operators and convergence can still be guaranteed so long as the errors satisfy a summability condition [23]. To use an indirect method we first reduce this system by elimination to

$$\begin{aligned} x &= (I + P + A^T A)^{-1}(\mu_x - A^T \mu_y), \\ y &= \mu_y - Ax, \end{aligned}$$

and note that the matrix  $I + P + A^T A$  is positive definite. This system is then solved with conjugate gradients (CGs) or a similar method [54, 56]. One iteration of CG requires multiplications with the matrices  $P$ ,  $A$ , and  $A^\top$ . If these matrices are very sparse, or fast multiplication routines exist for them, then one CG step can be very fast. We run CG until the residual satisfies an error bound, at which point we return the approximate solution. We can use techniques, such as warm-starting CG from the previous solution and using a preconditioner to improve the convergence [12].

**6.2. Cone projection.** Most convex optimization problems of interest can be expressed using a combination of the “standard” cones, namely the positive orthant, second-order cone, semidefinite cone, and exponential cone [52, 51]. These cones all have well-known projection operators [60]. Of these, only the semidefinite cone projection provides a computational challenge since it requires an eigen-decomposition, which may be costly. If our problem consists of the Cartesian product of many of these cones, then each of these projections can be carried out independently and in parallel.

Alternatively, since the cone projection step is totally separated from the rest of the algorithm, we can incorporate any number of problem-specific cones with their own projection operators, which may perform better in practice than reformulating the problem to use the standard cones. The restriction that the set be a cone is not too stringent, because we can write many convex constraints as a combination of a conic constraint and an affine constraint. In particular the set defined by a convex function  $f$  can be transformed as

$$\{s \mid f(s) \leq 0\} \Rightarrow \{(t, s) \mid tf(s/t) \leq 0, t \geq 0\} \cap \{(t, s) \mid t = 1\},$$

which is a combination of a convex cone and an affine equality constraint, which fits our framework. If the original convex set has an efficient projection operation, then in the worst case we can perform a bisection search over  $t \geq 0$  using the projection operator as a subroutine. In most cases the dominant cost of Algorithm 5.1 will be



solving the linear system, so the additional cost of a bisection to compute the cone projection will typically be negligible. As an example, consider the “box” cone defined as

$$\mathcal{K}_{\text{box}} = \{(t, s) \mid tl \leq s \leq tu, t \geq 0\},$$

where  $l, u \in \mathbb{R}^d$  are data. When combined with the constraint that  $t = 1$  this represents box constraints on the variable  $s$ , which is commonly used in LP and QP solvers. Projection onto this cone can be done via Newton’s method on the scalar variable  $t$ , which typically only requires a few iterations to reach convergence. This cone is supported in the SCS v3.0 solver.

**7. Numerical experiments.**

**7.1. Comparing Algorithm 5.1 to (5.2).** Here we compare the computational efficiency of using DR splitting applied to the homogeneous embedding (Algorithm 5.1) and DR splitting applied directly to the original problem (equation (5.2)) on a range of synthetic problems. We constructed feasible, primal infeasible, and unbounded (dual infeasible) QCPs over the positive orthant and compared the number of iterations taken by (5.2) with infeasibility detection using successive iterates and Algorithm 5.1. Since the cost per iteration is essentially identical for both approaches the number of iterations determines the overall solve time. The results on diverging sequences producing infeasibility certificates from Banjac et al. [5] and Liu, Ryu, and Yin [38] do not immediately carry over to the case of (5.2) since they only hold for ADMM applied to convex functions, and the matrix  $M$  is not the subdifferential of a convex function. That being said, we can still use the techniques and compare the performance in practice. In the next section we shall compare solvers that do come with theoretical guarantees.

We randomly generated 1000 feasible, infeasible, and unbounded problems of size  $n = 100$  and  $m = 150$ . For feasible problems we declared the problem to be solved when the maximum  $\ell_\infty$ -norm KKT violation was  $10^{-6}$ . Similarly, for infeasible and unbounded problems we stopped when the algorithms produced a valid certificate with  $\ell_\infty$ -norm tolerance of  $10^{-6}$ . For each problem we computed the ratio of the number of iterations required by (5.2) to the number required by Algorithm 5.1 to solve the problem or certify infeasibility. A higher ratio indicates that Algorithm 5.1 requires fewer iterations to solve the problem than (5.2). We present histograms of the performance ratio in Figures 1(a), 1(b), and 1(c) for feasible, infeasible, and unbounded problems, respectively. Evidently, generating certificates from the homogeneous embedding can be orders of magnitude faster; the geometric mean of the ratio on infeasible problems was 49.0 and on unbounded problems was 299.1. In fact our approach was not slower

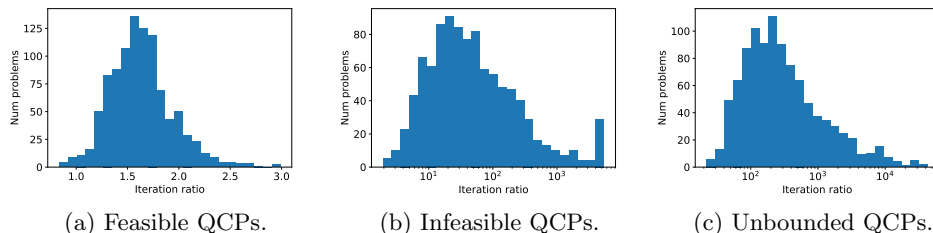


FIG. 1. Histograms of iteration count ratio of (5.2) to Algorithm 5.1. Higher ratios indicate that our approach is taking fewer iterations to reach the same accuracy.

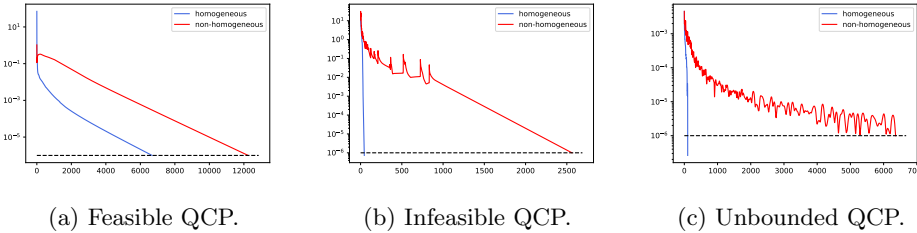


FIG. 2. Trace of max residuals under (5.2) (nonhomogeneous) and Algorithm 5.1 (homogeneous) for randomly selected example problems.

on a single instance. The successive differences approach failed to find a certificate of infeasibility within the iteration limit of  $10^5$  in 27 problems. For feasible problems the approach based on the homogeneous embedding is often quicker to find a solution, sometimes by a significant factor. The geometric mean of the ratios was 1.6, and the homogeneous embedding approach was faster in 987 of the 1000 problems.

In Figure 2 we show how the maximum  $\ell_\infty$ -norm residuals converge on randomly selected feasible, infeasible, and unbounded problems. For the feasible problem we plot the maximum KKT condition residual and for the infeasible and unbounded problems we plot the maximum residual from a valid certificate. For infeasible and unbounded problems the approach based on the homogeneous embedding converges to a certificate extremely rapidly, but the approach based on diverging iterates takes many iterations to produce a certificate. For the feasible problem the difference is less stark, but Algorithm 5.1 still converges faster, reaching the tolerance in about half the number of iterations required by (5.2).

**7.2. Comparing open-source solvers.** In this section we compare SCS v3.0, our open-source implementation of Algorithm 5.1 for QCPs, to other available open-source solvers that apply ADMM directly to QCPs. In particular we compare to OSQP [68] and COSMO [26], both of which rely on diverging iterates to generate certificates of infeasibility.

As discussed in section 5.4 the iterates produced by SCS v3.0 always satisfy the cone membership and complementarity KKT conditions defined in (3.10). Therefore to say that a problem is solved we need to check if the primal residual, dual residual, and duality gap are all below a certain tolerance. Specifically, SCS v3.0 terminates when it has found  $x \in \mathbb{R}^n$ ,  $s \in \mathbb{R}^m$ , and  $y \in \mathbb{R}^m$  that satisfy

$$\begin{aligned} \|Ax + s - b\|_\infty &\leq \epsilon_{\text{abs}} + \epsilon_{\text{rel}} \max(\|Ax\|_\infty, \|s\|_\infty, \|b\|_\infty), \\ \|Px + A^\top y - c\|_\infty &\leq \epsilon_{\text{abs}} + \epsilon_{\text{rel}} \max(\|Px\|_\infty, \|A^\top y\|_\infty, \|c\|_\infty), \\ |x^\top Px + c^\top x + b^\top y| &\leq \epsilon_{\text{abs}} + \epsilon_{\text{rel}} \max(|x^\top Px|, |c^\top x|, |b^\top y|), \end{aligned}$$

where  $\epsilon_{\text{abs}} > 0$  and  $\epsilon_{\text{rel}} > 0$  are user defined quantities that control the accuracy of the solution. For the purposes of our experimental results we set  $\epsilon_{\text{abs}} = 10^{-3}$  and  $\epsilon_{\text{rel}} = 10^{-4}$ . OSQP and COSMO have analogous quantities for the primal and dual residual; however, they do not allow the user to specify a bound on the gap. Therefore, in order to ensure that the gap is below the desired tolerance we solve each problem with these solvers with the initial choices of  $\epsilon_{\text{abs}}$  and  $\epsilon_{\text{rel}}$  and check if the gap is below the tolerance. If it is, then we return that solution; otherwise we halve  $\epsilon_{\text{abs}}$  and  $\epsilon_{\text{rel}}$  and re-solve. This procedure is continued until the solver returns a solution

that satisfies the gap constraint, and only the last solve counts toward the statistics. For a concrete case of why this is necessary take the BOYD2 problem from the Maros–Mészáros QP dataset. OSQP returns the certificate “solved” for this problem after 24300 iterations with an “optimal” objective of 343.32. However, the true optimal objective value for this problem is 21.26 [44]. The issue is that the duality gap of the primal-dual pair returned by OSQP is  $1.3 \times 10^3$ , when the desired gap is on the order of  $10^{-2}$ . Since the primal and dual residuals are small but the duality gap is large it means that OSQP has returned a primal-dual pair that is (almost) feasible but is far from optimal. On the other hand SCS v3.0, which terminates only when the gap as well as the primal and dual residuals are below the tolerance, returns a solution after 3250 iterations with an objective value of 21.12, significantly closer to the true value.

Since the cone memberships are always guaranteed by the iterates, SCS v3.0 declares a problem infeasible when it finds  $y \in \mathbb{R}^m$  that satisfies

$$b^\top y = -1, \quad \|A^\top y\|_\infty < \epsilon_{\text{infeas}}.$$

Similarly, SCS v3.0 declares dual infeasibility when it finds  $x \in \mathbb{R}^n$ ,  $s \in \mathbb{R}^m$  that satisfy

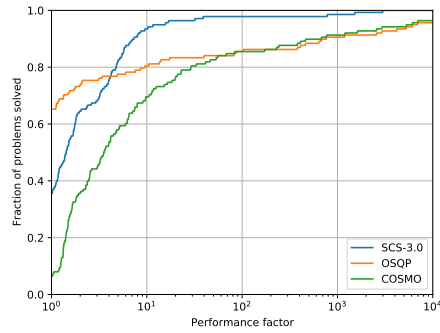
$$c^\top x = -1, \quad \max(\|Px\|_\infty, \|Ax + s\|_\infty) < \epsilon_{\text{infeas}}.$$

The other solvers have analogous certificates, and in these cases there is no duality gap so the iterative procedure is not required. For the experiments we set  $\epsilon_{\text{infeas}} = 10^{-4}$ .

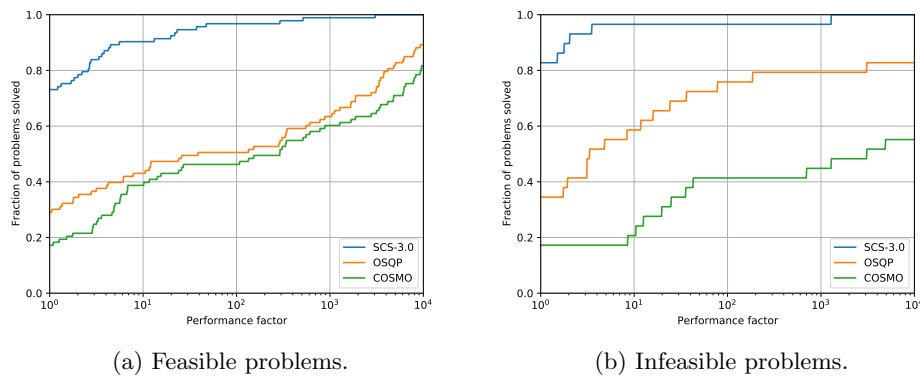
All three solvers rescale the data to yield better conditioning and they all implement a heuristic “step-size” adaptation scheme. These heuristics were enabled for these experiments; however, we note that the conclusions we derive from the experiments did not change when these heuristics were disabled. On the contrary, the advantage that the homogeneous embedding had over the direct approaches was more pronounced in that case. We disabled more advanced techniques like acceleration, solution polishing, and semidefinite cone decomposition. All three solvers were given a maximum iteration limit of  $10^5$  and a time-limit of  $10^3$  seconds per problem. If a solver fails to find a solution or a certificate of infeasibility satisfying the tolerances within those limits, then it is considered to have failed to solve that problem. When measuring average run-times any failures are assigned the maximum run-time of  $10^3$  seconds. All experiments were run single-threaded on a 2017 MacBook pro with a 3.1GHz Intel i7 and 16GB of RAM.

We present results on several datasets. First we present results on the Maros–Mészáros dataset of challenging convex feasible QPs [44], then on the NETLIB dataset, which contains both feasible and infeasible linear programs [27]. The SDPLIB dataset also has four infeasible problems, on which we test SCS v3.0 and COSMO [8] (OSQP does not support the semidefinite cone). Finally, we present results on randomly generated quadratic problems as in the previous section. To summarize the results for each dataset we shall use Dolan–Moré performance profiles [19]. In these plots each point of the curve corresponds to what fraction of the problems are solved ( $y$ -axis) within a factor ( $x$ -axis) of the fastest solver for each problem. Curves of faster solvers appear above those of slower solvers. When summarizing wall-clock performance we shall use the shifted geometric means of the run-times with a shift of 10 seconds, denoted `sgm10`.

In Figure 3 we show the Dolan–Moré profile for the Maros–Mészáros QP dataset and in Table 2 we present the failure rates. SCS v3.0 is the most robust solver, with around a third of the failures of the next best solver. In terms of solve speeds SCS

FIG. 3. *Performance profiles for Maros–Mészáros problems.*TABLE 2  
*Solver failure rates on Maros–Mészáros problems.*

SCS-3.0	OSQP	COSMO
5.80%	18.12%	16.67%



(a) Feasible problems.

(b) Infeasible problems.

FIG. 4. *Performance profiles for NETLIB LP problems.*

v3.0 was the fastest, followed by COSMO, which was about  $2.6\times$  slower, and then OSQP, which was about  $2.8\times$  slower, as measured by `sgm10`.

In Figure 4 we show the profiles for the NETLIB dataset, broken down into feasible and infeasible problems. In this case it is clear that SCS v3.0 is the fastest solver. This is partially explained by the fact that SCS v3.0 appears to be far more robust for these problems with a significantly lower overall failure rate, as shown in Tables 3 and 4. For the feasible problems SCS v3.0 was about  $16\times$  faster than OSQP and  $20\times$  faster than COSMO as measured by `sgm10`. For the infeasible problems SCS v3.0 was about  $3.2\times$  faster than OSQP and  $20\times$  faster than COSMO.

The results for all four infeasible SDPLIB problems are given in Table 5. Both SCS v3.0 and COSMO successfully certified that these problems were infeasible (primal or dual depending on the problem), but SCS v3.0 was able to certify infeasibility significantly faster than COSMO, about  $66\times$  faster in terms of `sgm10`. This difference

TABLE 3  
*Solver failure rates on NETLIB infeasible problems.*

SCS-3.0	OSQP	COSMO
20.69 %	37.93%	75.86%

TABLE 4  
*Solver failure rates on NETLIB feasible problems.*

SCS-3.0	OSQP	COSMO
12.90 %	61.29%	65.59%

TABLE 5  
*Solver times on SDPLIB infeasible problems in seconds.*

	SCS-3.0	COSMO
infd1	0.0122	2.1776
infd2	0.0155	0.0321
infp1	0.0035	0.1154
infp2	0.0037	0.1119

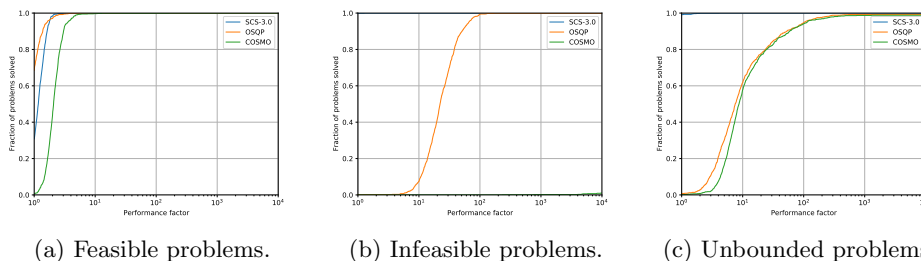


FIG. 5. *Performance profiles for randomly generated QP problems.*

is partially explained by the number of iterations required to generate a certificate. COSMO required almost  $10\times$  the number of iterations of SCS v3.0 to certify that these problems were infeasible.

Finally, the results for the random QPs are presented in Figure 5. For feasible random problems SCS v3.0 and OSQP have similar performance, with OSQP about 4% faster than SCS v3.0 on average, and COSMO somewhat slower. All three solvers solved all feasible instances. However, for the randomly generated infeasible and unbounded problems the difference is stark. For unbounded problems SCS v3.0 certified every single problem correctly, OSQP had a 0.8% failure rate, and COSMO had a 1.2% failure rate. However, SCS v3.0 was about  $32\times$  faster than OSQP and  $41\times$  faster than COSMO, as measured by `sgm10`. For the infeasible problems again SCS v3.0 was able to certify infeasibility on all problems and OSQP on all but one problem, but COSMO was unable to certify infeasibility on even a single instance, hitting the maximum iteration limit on every problem. Even when the infeasibility tolerances were loosened COSMO still struggled. This explains the strange Dolan–Moré profile

for this problem set, where SCS v3.0 is barely visible at the top left and COSMO barely visible in the bottom right. Even though OSQP and SCS v3.0 had similar success rates, SCS v3.0 was able to certify infeasibility about  $29\times$  faster as measured by `sgm10`.

**8. Conclusion.** We applied Douglas–Rachford splitting to a homogeneous embedding of the linear complementarity problem (LCP). This resulted in a simple alternating procedure in which we solve a linear system and project onto a cone at each iteration. Since the linear system does not change from one iteration to the next we can factorize the matrix once and cache it for use thereafter. Our procedure is able to return the solution to the LCP when one exists or a certificate of infeasibility otherwise. Quadratic cone programs (QCPs) are an important special case of LCPs and we discussed how to implement the procedure efficiently for QCPs in detail. We concluded with some experiments demonstrating the advantage of our procedure over competing approaches numerically, showing large speedups for infeasible problems without sacrificing performance on feasible problems. The algorithm has been implemented in C and is available as an open-source QCP solver.

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